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Quantum radiation from moving dielectrics in two, three, and more spatial dimensions

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Abstract. Quantum radiation from a moving dielectric semi-infinite half-space is investigated in a scalar model in two, three, and arbitrary integer spatial dimensions. The dielectric medium is non-dispersive and non-absorbing. The Hamiltonian describing the field in the presence of the moving dielectric body is obtained from considerations of Lorentz invariance. The field is quantized by an expansion into photon eigenmodes that are composed of incoming, reflected, and transmitted plane waves. The treatment includes total internal reflection and evanescent modes which are specific to dimensions $d \geqslant 2$. The Schrödinger equation for the state of the field is solved to first order in the velocity of the dielectric by a generalized adiabatic approximation. Non-uniform motion is found to lead to the emission of photon pairs whose spectrum and angular distribution are analysed in detail. The radiated energy and the radiation-reaction force are studied with particular emphasis to their dependence on the dimension of the system. In addition, the energy is worked out for several test trajectories.

1. Introduction

1.1. State of the art and motivation

Although the fact that moving mirrors radiate has been known for more than 20 years [1], and the effect has long since found its way into textbooks [2] as the Unruh effect, the underlying physical mechanism is still viewed by many as somewhat mysterious, largely because most investigations have been done for highly idealized models. Almost all previous workers‡ have made the sometimes convenient but necessarily unphysical assumption of perfect reflectivity of the mirror which then leads to the mathematically challenging problem of finding solutions to the wave equation with time-dependent boundary conditions. Albeit difficult, finding such solutions is not impossible. In a pioneering paper Moore [3] showed that exact solutions can be found for the wave equation in one (spatial) dimension. However, it has meanwhile been recognized [4] that for arbitrary motions this is possible only in one dimension because only the one-dimensional wave equation remains invariant under conformal transformations once boundary conditions have been imposed.

The restriction to perfect reflectivity of the mirror has been overcome by a few workers. One approach [5] is to describe an imperfect mirror by defining its properties of reflectivity

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‡ Here we dispense with a complete bibliography of works on radiation by moving perfect mirrors, as there exists a vast number. A skeleton of essential references can be found in [6].

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and transmittivity and to consider the scattering of photons from this mirror; this amounts to dealing with linear response theory which is valid for small displacements of the mirror. Another possibility is to construct an electromagnetic field theory in the presence of a non-dispersive dielectric medium, which has been done both for a one-dimensional scalar model [6] and for the three-dimensional electromagnetic field [8]. The latter approach has the unique advantage that it gives full information on the state of the photon field and hence the spectrum of the emitted photons. In a recent paper Barton and North [9] proceed from this concept for calculating the radiated energy from an harmonically oscillating dielectric half-space interacting with a scalar quantum field in three dimensions. This paper uses the same approach but solves a more general problem. Here we consider a scalar field in two and three dimensions and even in arbitrarily many dimensions $d > 1$. The primary aim of this is to show that two- and more dimensional fields interacting with moving mirrors exhibit a qualitatively different behaviour from one-dimensional models. The basic reason for this is that imperfect reflection in more than one dimension always comprises total internal reflection and therefore evanescent field modes, and these lead to qualitatively new features of the system. Furthermore, unlike Barton and North we do not restrict ourselves to purely oscillatory motions, but instead consider arbitrary non-relativistic motions normal to the reflecting surface and starting from rest. This gives important new information on the dependence of the dissipative force acting on the moving mirror on the dimensionality d of the system and, in particular, shows that there is a significant difference between even and odd dimensions caused by a different behaviour under time-reversal. In addition, we are able to examine the radiated energy from a moving mirror in arbitrary dimensions and come to the surprising conclusion that for dimensions greater than three the radiated energy diverges in the perfect-reflector limit, which points to a fundamental flaw in the physics of any models considering moving perfect mirrors in more than three dimensions. Moreover, we are presenting this analysis with the intention of illustrating the precise mechanism of the radiation. We are for the first time able to show exactly how much radiation is going where and how this depends on the refractivity of the medium. Finally, the examination of a variety of qualitatively different examples for possible trajectories of the mirror puts us in the position to make some general assertions on the dependence of the emitted radiation on the motion of a perfectly or imperfectly reflecting mirror.

1.2. Outline

After we have, in the remainder of this section, given a precise definition of the model we investigate in this paper and of the notation we use, we establish a canonical field theory of a quantized scalar field in the presence of a dielectric half-space in section 2. In section 3 we solve the time-dependent Schrödinger equation for the state of the field in a suitably adapted version of the standard adiabatic approximation. We work to first order in the velocity of the dielectric body, i.e. we adopt a non-relativistic approximation. Once we have determined the time evolution of the state vector of the field, we investigate in section 4 the spectral distribution of the quantum radiation that is produced by the motion of the dielectric. In section 5 we calculate the total radiated energy and the radiation-reaction force on the dielectric body and investigate the effects of the dimensionality of the system on these quantities. The paper closes with a brief summary of the results in section 6.

1.3. The model

The object of this paper is the investigation of the interaction of a quantized massless scalar field in its ground state with a non-relativistically moving dielectric half-space. Although this work is mainly concerned with fields in two and three spatial dimensions, we formulate the theory for an arbitrary number of spatial dimensions d. Hence all mathematical notations that depend on the spatial dimension, e.g. the position vector $\boldsymbol{r} = (x_1, x_2, \ldots, x_d)$ or volume integrals $\int \mathrm{d}^d\boldsymbol{r} \equiv \int \mathrm{d}\boldsymbol{r}$, are to be interpreted accordingly. Even though we have no direct physical interpretation of systems with spatial dimensions greater than three, the generality of our approach is justified by providing a deeper insight into the dependence of the radiation effects studied here on the spatial dimensionality of the system.

The restriction to a scalar as opposed to a vector field has been made to avoid the bulk of technicalities that comes with having to distinguish several different polarizations. However, there is no conceptional difficulty that would impede the investigation of the full Maxwell field†. In fact, for the very special case of a spherical cavity with varying radius the radiation of real transverse electromagnetic photons has already been studied by the same approach as presented here [8].

For simplicity, we presently consider only a semi-infinite dielectric half-space, i.e. a dielectric body that has just a single interface with empty space. Of course, this is in a strict sense a pathological situation, but it nevertheless captures all essential physics of the effects we wish to consider. Calculations for finitely extended dielectric bodies, which have at least two interfaces, would be considerably more complicated because the normal modes of the field would then have to include multiple reflections and refractions, and in a formalism of canonical quantization this leads to severe difficulties since the completeness of the mode functions requires infinitely many reflections and refractions to be taken into account.

The complex scalar field $\phi(\boldsymbol{r}, t)$ that describes our field obeys the wave equation

$$\boldsymbol{\nabla}^2\phi(\boldsymbol{r}, t) - \varepsilon(\boldsymbol{r})\frac{\partial^2}{\partial t^2}\phi(\boldsymbol{r}, t) = 0 \tag{1.1}$$

where $\varepsilon(\boldsymbol{r})$ denotes the position-dependent dielectric response function. We use the word 'dielectric' in loose analogy to the Maxwell field; one can think of the derivatives $-(\partial\phi/\partial t)$ and $\boldsymbol{\nabla}\phi$ as loosely analogous to the Maxwell $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. The physical requirement on the field ϕ to be finite everywhere, implies that ϕ itself and its first-order spatial derivatives must be continuous everywhere. Thus the field ϕ is subjected to the continuity conditions

$$\phi \text{ and } \frac{\partial\phi}{\partial x_i} \qquad \text{continuous for } i = 1, 2, \ldots, d. \tag{1.2}$$

We define the dielectric half-space to consist of a non-absorbing, non-magnetic, and non-dispersive polarizable medium, which can be modelled through a real-valued space-dependent dielectric response function. This description is nothing more than a macroscopic characterization of the medium at any given point in space through its (uniform) response to an externally applied field. Within this model the medium itself is assumed to be strictly

† In fact, the transverse electric (TE) and transverse magnetic (TM) modes of the electromagnetic field behave essentially just like two decoupled scalar fields if, like in this paper, only motions normal to the interface of the dielectric are considered. In this case the translation invariance and complete isotropy of the system parallel to the surface prevent the creation of photon pairs with mixed TE and TM polarizations from the vacuum because of parity conservation. Mathematically this is easily appreciated from the vacuum-to-two-photon matrix elements of the force (cf e.g. Barton [7]) which are the main ingredients in the photon pair production amplitude (3.5). An explicit calculation of several aspects of TE and TM quantum radiation by a three-dimensional moving dielectric half-space has been done by North [7].

neutral and unpolarized. Any microscopic features or polarizations not induced by external fields lie beyond the present approach. We also stress that our model treats the dielectric medium as a perfectly rigid body. This is, of course, a very crude approximation to any realistic physical media, but it nevertheless captures the essential physics of the effects we wish to investigate (and that is why it has likewise been made by all previous workers in the field). Quantum radiation from compressible dielectrics introduces a whole range of new effects that are beyond this study and are considered elsewhere [10].

The semi-infinite dielectric half-space then corresponds to a discontinuous step in the dielectric response function of the system space. We choose the spatial Cartesian coordinate system defined by the orthonormal unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d$ such that the normal vector $\boldsymbol{n}$ of the surface of the medium coincides with $\boldsymbol{e}_1$. We define $\boldsymbol{n}$ to be pointing into the medium. Thus the interface plane spanned by $\boldsymbol{e}_2, \ldots, \boldsymbol{e}_d$ lies perpendicular to the x_1-axis with the medium extending in the positive x_1-direction.

With these definitions in place, we can characterize the instantaneous spatial configuration of the system by the time-dependent x_1-coordinate of the position of the interface which we will denote by $\xi(t)$. The spatial dielectric response function of the system-space then reads

$$\varepsilon(\boldsymbol{r}, t) = \Theta(\xi(t) - x_1) + n^2\Theta(x_1 - \xi(t)) \tag{1.3}$$

where $n > 1$ denotes the refractive index of the medium, and Θ is the Heaviside step function. According to what we said earlier, n is a constant both in time and in frequency; it is a merely measure for the strength of the response of the medium to external fields. We note in passing that the dielectric half-space described by the response function of equation (1.3) is invariant under lateral translations, which will considerably simplify some calculations of this work.

We restrict the present investigation to velocities $\boldsymbol{\beta}(t)$ of the medium normal to its interface; hence the velocity $\boldsymbol{\beta}(t)$ of the interface is given by

$$\boldsymbol{\beta} = (\beta, 0, \ldots, 0) \qquad \text{with } \beta(t) = \frac{\partial}{\partial t}\xi(t). \tag{1.4}$$

Throughout our calculation we assume that the velocity is non-relativistic, i.e. that $\beta \ll c$.

The description of the eigenmode functions of the field in the present formalism requires a characterization of plane waves with respect to the interface of the dielectric half-space. Since by definition this interface lies perpendicular to the x_1-direction, we define the surface component $\boldsymbol{k}_\parallel$ of the wavevector $\boldsymbol{k}$ of a plane wave by

$$\boldsymbol{k}_\parallel = \sum_{j=2}^{d} k_j \boldsymbol{e}_j. \tag{1.5}$$

Then, except in the trivial case of normal incidence when $\boldsymbol{k}_\parallel = 0$, the normal vector $\boldsymbol{n}$ of the interface and the surface component $\boldsymbol{k}_\parallel$ of a wavevector $\boldsymbol{k}$ are orthogonal and define a two-dimensional plane of incidence of a plane wave.

If the plane of incidence of a plane wave is given, i.e. if the surface component $\boldsymbol{k}_\parallel$ of its wavevector is known, it is possible to characterize the plane wave by the angle of incidence $\alpha_{\boldsymbol{k}} \in [0, \pi]$ which represents the angle that is enclosed by its wavevector $\boldsymbol{k}$ and the normal vector $\boldsymbol{n}$.

$$k \cos\alpha_{\boldsymbol{k}} = \boldsymbol{k} \cdot \boldsymbol{n} \qquad \text{with } \alpha_{\boldsymbol{k}} \in [0, \pi]. \tag{1.6}$$

We stress that, according to this definition, $\alpha_{\boldsymbol{k}}$ is not identical to the angle of incidence which is known from usual textbook treatments. Being commonly defined as the deviation from normal incidence the standard angle of incidence is restricted to the interval [0, $(\pi/2)$] and thus does not normally carry any information about from which side a plane wave approaches the interface. It is therefore not suited for a characterization of the wavevector $\boldsymbol{k}$ in the present context.

Throughout this paper partial derivatives with respect to spatial coordinates will be abbreviated according to

$$\frac{\partial}{\partial x_j}\phi \equiv \phi_{x_j}. \tag{1.7}$$

CGS units are used everywhere in the paper; $\hbar$ and c are set equal to 1 unless explicitly indicated. All special functions are defined as in [11, 12].

2. The canonical theory of the field

A scalar field whose equation of motion is given by the wave equation (1.1) with time-independent $\varepsilon(\boldsymbol{r})$ can be described by the Lagrangian density

$$\mathcal{L}_0 = \tfrac{1}{2}(\varepsilon(\boldsymbol{r};\xi)\dot{\phi}^2 - (\nabla\phi)^2) \tag{2.1}$$

where we have introduced the subscript 0 to label quantities that describe the system in the rest-frame of the dielectric body. Correspondingly, the conjugate momentum π of the field ϕ and the Hamiltonian H_0 in the rest-frame of the dielectric are given by

$$\pi = \varepsilon\dot{\phi} \tag{2.2}$$

and

$$H_0 = \int \mathrm{d}\boldsymbol{r}\,\frac{1}{2}\left(\frac{\pi^2}{\varepsilon(\boldsymbol{r};\xi)} + (\nabla\phi)^2\right) \tag{2.3}$$

respectively.

If the dielectric moves then the dielectric response function ε in the wave equation (1.1) becomes time-dependent as indicated in equation (1.3) and the above Lagrangian and Hamiltonian no longer describe the system, i.e. their equations of motion no longer coincide with the wave equation. The Lagrangian for a uniformly moving medium can be found from considerations of Lorentz invariance. It turns out† that the Lagrangian density for a moving dielectric is uniquely determined by the three basic requirements: (i) that it is a true Lorentz scalar, (ii) that in the limit $\varepsilon = 1$ it reduces to the Lagrangian density $\mathcal{L}_0(\varepsilon = 1)$ for the fields in vacuum, and (iii) that in the limit $\beta = 0$ it turns into the Lagrangian density $\mathcal{L}_0$ for the fields in a stationary dielectric. In this way we find

$$\mathcal{L} = \tfrac{1}{2}\{(\varepsilon - 1)(v^\mu\partial_\mu\phi)^2 + (\partial^\mu\phi)(\partial_\mu\phi)\} \tag{2.4}$$

where v is the $(d+1)$-dimensional velocity vector in Minkowski space

$$v = \frac{1}{\sqrt{1-\beta^2}}(1, \boldsymbol{\beta}). \tag{2.5}$$

The canonical formalism then yields the Hamiltonian

$$H = H_0 + \Delta H \tag{2.6}$$

$$\Delta H = \boldsymbol{\beta}\cdot\boldsymbol{P} = \boldsymbol{\beta}\cdot\int \mathrm{d}\boldsymbol{r}\,\frac{(1-\varepsilon)}{\varepsilon}\pi(\nabla\phi). \tag{2.7}$$

† The method of finding the Lagrangian and the Hamiltonian for a dielectric in uniform motion has been discussed in detail in appendix B of [6], in appendix A of [8], and in section 2.2 of [13].

Here we have already carried out an expansion in powers of β and neglected any terms of order β^2 and higher. In other words, $\Delta\mathcal{H}$ is a first-order perturbation. This is all we need since we aim at first-order perturbation theory in the small parameter β, which is wholly adequate for this problem as all material motion is slow compared with the speed of light.

In the following we will use the above Hamiltonian irrespective of whether the motion of the dielectric is uniform, even though this is what we have assumed in its derivation. Hence we ignore all acceleration stresses, which is a viable approximation as long as the body can be considered rigid [6].

Since the force exerted on the dielectric body by the field is an essential characteristic of the radiation process, we proceed by introducing the expression for the force $\boldsymbol{F}_0$ in the rest-frame of the dielectric. This force can be determined either by calculating the force acting on a charge distribution induced by the field inside the dielectric, or by taking the time-derivative of the momentum flux of the field [6, 8, 13]. Taking the result from [13], we quote the x_1-component of the force $\boldsymbol{F}_0$ which is given by a surface integral over the interface of the half-space

$$F_0 = \frac{(n^2-1)}{2n^2}\int \mathrm{d}x_2 \ldots \int \mathrm{d}x_d \left\{\phi_{x_1}^2 - \sum_{j=2}^{d}\phi_{x_j}^2\right\}_{(x_1=\xi)}. \tag{2.8}$$

Higher-order corrections for moving dielectrics are not needed in our first-order calculation.

In order to quantize the system we demand standard equal-time commutation relations for the field operators ϕ and π and expand them into normal modes in terms of photon annihilation and creation operators, $a_{\boldsymbol{k}}(\xi)$ and $a_{\boldsymbol{k}}^{\dagger}(\xi)$, so that the Hamiltonian (2.3) is diagonalized to $H_0 = \int \mathrm{d}\boldsymbol{k}\,\omega_{\boldsymbol{k}}/2\,(a_{\boldsymbol{k}}a_{\boldsymbol{k}}^{\dagger} + a_{\boldsymbol{k}}^{\dagger}a_{\boldsymbol{k}})$ by the expansion (see [14] for details); we write

$$\phi(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{k}\,\frac{1}{\sqrt{(2\pi)^d 2\omega_{\boldsymbol{k}}}}\{a_{\boldsymbol{k}}(\xi) f_{\boldsymbol{k}}(\boldsymbol{r};\xi) + a_{\boldsymbol{k}}^{\dagger}(\xi) f_{\boldsymbol{k}}^{*}(\boldsymbol{r};\xi)\} \tag{2.9a}$$

$$\pi(\boldsymbol{r}) = \varepsilon(\boldsymbol{r};\xi)\int \mathrm{d}\boldsymbol{k}\,\frac{-\mathrm{i}\omega_{\boldsymbol{k}}}{\sqrt{(2\pi)^d 2\omega_{\boldsymbol{k}}}}\{a_{\boldsymbol{k}}(\xi) f_{\boldsymbol{k}}(\boldsymbol{r};\xi) - a_{\boldsymbol{k}}^{\dagger}(\xi) f_{\boldsymbol{k}}^{*}(\boldsymbol{r};\xi)\} \tag{2.9b}$$

where ξ is arbitrary but fixed, and the mode functions $f_{\boldsymbol{k}}$ are solutions of the wave equation (1.1) and obey the continuity conditions (1.2). Similarly to the approach of [15] we decompose each individual mode into three plane wave components which correspond to incoming, reflected, and transmitted waves and are characterized by the three wavevectors $\boldsymbol{k}^{\mathrm{in}}$, $\boldsymbol{k}^{\mathrm{re}}$, and $\boldsymbol{k}^{\mathrm{tr}}$. The expressions for the mode functions are then uniquely determined by the imposition of the continuity conditions (1.2); we find [13]:

$$\begin{aligned}
f_{\boldsymbol{k}}^{+}(\boldsymbol{r};\xi) &= \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{r}_{\parallel}}\begin{cases} \mathrm{e}^{\mathrm{i}k_1(x_1-\xi)} + R_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}k_1(x_1-\xi)} & \text{if } x \leqslant \xi \\ T_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}k_1^{\mathrm{tr}}(x_1-\xi)} & \text{if } x \geqslant \xi \end{cases} \\
f_{\boldsymbol{k}}^{-}(\boldsymbol{r};\xi) &= \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{r}_{\parallel}}}{n^2}\begin{cases} T_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}k_1^{\mathrm{tr}}(x_1-\xi)} & \text{if } x \leqslant \xi \\ \mathrm{e}^{\mathrm{i}k_1(x_1-\xi)} + R_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}k_1(x_1-\xi)} & \text{if } x \geqslant \xi \end{cases}
\end{aligned} \tag{2.10}$$

where the label $\boldsymbol{k}$ is the wavevector of the incoming plane wave component, i.e. $\boldsymbol{k} \equiv \boldsymbol{k}^{\mathrm{in}}$, and, as it has been introduced in the previous section, ξ denotes the x_1 coordinate of the location of the dielectric interface. The reflection and transmission coefficients are given by

$$R_{\boldsymbol{k}} = \frac{k_1 - k_1^{\mathrm{tr}}}{k_1 + k_1^{\mathrm{tr}}} \qquad \text{and} \qquad T_{\boldsymbol{k}} = \frac{2k_1}{k_1 + k_1^{\mathrm{tr}}} \tag{2.11}$$

respectively.

The superscripts + and − of the eigenmode functions refer to the sign of the k_1 component of the wavevector $\boldsymbol{k}$. This notation discriminates between the eigenmodes $f_{\boldsymbol{k}}^{+}$ that have the incoming and reflected wave components propagating in vacuum and the transmitted wave component propagating in the medium, and those $f_{\boldsymbol{k}}^{-}$ where the opposite holds. Since total internal reflection occurs only for $f_{\boldsymbol{k}}^{-}$, the reflection and refraction processes are qualitatively very different in the two directions, so that the discrimination (±) is crucial in the expressions to follow. For compactness we absorb the discrimination of these two cases into the two auxiliary functions $\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}$ and $\varepsilon_{\boldsymbol{k}}^{\mathrm{tr}}$, which are defined such that $\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}$ takes the value of the dielectric response function in the half-space of the incoming and reflected wave components of the eigenmode $f_{\boldsymbol{k}}^{\pm}$, and $\varepsilon_{\boldsymbol{k}}^{\mathrm{tr}}$ takes the value in the half-space of the transmitted wave component.

Expressed in terms of these auxiliary functions the relations between the frequency $\omega_{\boldsymbol{k}}$ and the wavevectors of an eigenmode function are given by

$$\omega_{\boldsymbol{k}}^2 = \frac{(\boldsymbol{k})^2}{\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}} = \frac{(\boldsymbol{k}^{\mathrm{re}})^2}{\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}} = \frac{(\boldsymbol{k}^{\mathrm{tr}})^2}{\varepsilon_{\boldsymbol{k}}^{\mathrm{tr}}}. \tag{2.12}$$

These equations, which follow directly from the wave equation (1.1), in turn lead to the relation between the k_1 components of the incoming and transmitted wavevectors $\boldsymbol{k}$ and $\boldsymbol{k}^{\mathrm{tr}}$

$$k_1^{\mathrm{tr}} = \frac{k_1}{|\cos\alpha_{\boldsymbol{k}}|}\sqrt{\frac{\varepsilon_{\boldsymbol{k}}^{\mathrm{tr}}}{\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}} - \sin^2\alpha_{\boldsymbol{k}}} \tag{2.13}$$

where $\alpha_{\boldsymbol{k}}$ is the angle of incidence which was introduced in equation (1.6). This expression shows that for eigenmodes with angles of incidence that fulfil $\sin^2\alpha_{\boldsymbol{k}} > \varepsilon_{\boldsymbol{k}}^{\mathrm{tr}}/\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}$, k_1^{tr} becomes purely imaginary. Since this can happen only if $\varepsilon_{\boldsymbol{k}}^{\mathrm{tr}}/\varepsilon_{\boldsymbol{k}}^{\mathrm{in}} < 1$, evanescent wave components can occur only for incident waves which approach the interface from the inside of the medium or, equivalently, modes that have $\alpha_{\boldsymbol{k}} \in [(\pi/2), \pi]$. Accordingly, we define the critical angle by

$$\sin^2\alpha_c = \frac{1}{n^2} \qquad \text{with } \alpha_c \in [(\pi/2), \pi] \tag{2.14}$$

so that the interval of angles $\alpha_{\boldsymbol{k}}$ that correspond to modes with complex-valued $\boldsymbol{k}^{\mathrm{tr}}$ vectors is given by $[(\pi/2), \alpha_c]$. Note that, as was mentioned before, α_c is not identical to the critical angle of incidence commonly defined in textbooks, which is restricted to $[0, (\pi/2)]$. By further introducing the auxiliary complex function $\mathcal{C}$ of the angle α

$$\mathcal{C}(\alpha) = \frac{1}{|\cos\alpha|}\begin{cases} \sqrt{n^2 - \sin^2\alpha} & \text{for } \alpha \in [0, (\pi/2)] \\ \mathrm{i}\sqrt{\sin^2\alpha - \dfrac{1}{n^2}} & \text{for } \alpha \in [(\pi/2), \alpha_c] \\ \sqrt{\dfrac{1}{n^2} - \sin^2\alpha} & \text{for } \alpha \in [\alpha_c, \pi] \end{cases} \tag{2.15}$$

which is either real or purely imaginary, we can rewrite equation (2.13) in the compact form

$$k_1^{\mathrm{tr}} = k_1\mathcal{C}(\alpha_{\boldsymbol{k}}). \tag{2.16}$$

We have defined $\mathcal{C}$ such that the wave described by $\boldsymbol{k}^{\mathrm{tr}}$ travels either in the same x_1-direction as the incoming wave described by $\boldsymbol{k}$ or is exponentially decaying into the vacuum.

It can further be shown (cf [13, 15, 16]) that for each arbitrary but fixed value of ξ the set of eigenmode functions $f_{\boldsymbol{k}}$ defined by (2.10) fulfils the relations of orthonormality and completeness given by

$$\int \mathrm{d}\boldsymbol{r}\, \varepsilon(\boldsymbol{r};\xi) f_{\boldsymbol{k}}(\boldsymbol{r};\xi) f^*_{\boldsymbol{k}'}(\boldsymbol{r};\xi) = (2\pi)^d \delta(\boldsymbol{k}-\boldsymbol{k}') \tag{2.17}$$

and

$$\int \mathrm{d}\boldsymbol{k}\, \sqrt{\varepsilon(\boldsymbol{r};\xi)} f_{\boldsymbol{k}}(\boldsymbol{r};\xi) \sqrt{\varepsilon(\boldsymbol{r}';\xi)} f^*_{\boldsymbol{k}}(\boldsymbol{r}';\xi) = (2\pi)^d \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{2.18}$$

respectively. These relations are essential for proving that the mode expansions (2.9a, b) diagonalize the Hamiltonian H_0 (2.3) and translate the field commutators into photon commutators.

We emphasize that the eigenmode functions $f_{\boldsymbol{k}}$ diagonalize only the Hamiltonian H_0 for the dielectric at rest and not the full Hamiltonian $H_0 + \Delta H$ that describes a moving dielectric. For our purposes of first-order perturbation theory this is sufficient. Beyond this approximation one faces severe difficulties even when attempting to diagonalize the complete Hamiltonian $H_0 + \Delta H$ in the seemingly simple case of constant velocities. This is because reflection from a moving interface causes the frequency of the reflected wave component to be Doppler shifted relative to the incoming and transmitted components, so that eigenmode functions, which have to be monochromatic because the frequency is their eigenvalue, can no longer be constructed from the physical principle of reflection and transmission. The diagonalization of $H_0 + \Delta H$ for arbitrary velocities $\beta(t)$ is of course even less feasible; it would correspond to an exact solution of the problem, which seems to be unattainable for more than one-dimensional systems, as explained in the introduction.

3. Photon emission in first-order perturbation theory

The emission of photons due to the normal motion of the dielectric half-space is described by the probability amplitudes of transitions of the photon field from its ground state into higher photon states. These transition amplitudes are determined by the time-dependent Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = [H_0(\xi(t)) + \Delta H(\xi(t), \beta(t))]|\psi(t)\rangle \tag{3.1}$$

with the initial condition $|\psi(t_0)\rangle = |0\rangle$ that the field is in its ground state at $t = t_0$. The solution of this equation is not immediately accessible by textbook methods. The Hamiltonian ΔH cannot be treated as an ordinary perturbation in standard perturbation theory because it as well as the unperturbed Hamiltonian H_0 depend on time through the parameters ξ and β. Conversely, the method of adiabatic approximation cannot be applied to the problem in a straightforward manner because only the eigenstates of H_0 are known but not those of the total parameter-dependent Hamiltonian $H_0 + \Delta H$. However, it has been shown in [8] that a perturbative solution of this equation can be found by combining these two standard approximation methods. By applying this generalized adiabatic approximation to the present system (cf [13]), we find that, to first-order in β, the initial state evolves into a superposition of the ground state $|0;\xi\rangle$ and two-photon states $|\boldsymbol{k}\boldsymbol{k}';\xi\rangle$

$$|\psi(t)\rangle = |0;\xi\rangle + \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}'\, c_{\boldsymbol{k},\boldsymbol{k}'} \mathrm{e}^{-\mathrm{i}(\omega_k+\omega_{k'})(t-t_0)} |\boldsymbol{k},\boldsymbol{k}';\xi\rangle \tag{3.2}$$

with the coefficients

$$c_{\boldsymbol{k},\boldsymbol{k}'} = \left\{ \frac{1}{\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'}} \langle \boldsymbol{k}, \boldsymbol{k}'; \xi | \frac{\partial H_0}{\partial \xi} | 0; \xi \rangle - \mathrm{i} \langle \boldsymbol{k}, \boldsymbol{k}'; \xi | P | 0; \xi \rangle \right\} \int_{t_0}^{t} \mathrm{d}\tau \, \beta(\tau) \mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})(\tau - t_0)}. \tag{3.3}$$

(P is the x_1 component of the vector $\boldsymbol{P}$ defined in equation (2.7).) Hence the transition amplitudes are $\langle \boldsymbol{k}\boldsymbol{k}' | \psi(t) \rangle = c_{\boldsymbol{k},\boldsymbol{k}'} \exp[-\mathrm{i}(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})(t - t_0)]$. We note that the matrix elements inside the curly brackets in (3.3), which are readily identified as distinct contributions from the adiabatic approximation and from standard time-dependent perturbation theory, are in fact independent of ξ due to the translation invariance of the system, so that the time-dependence of the coefficients $c_{\boldsymbol{k},\boldsymbol{k}'}$ enters only through the finite-interval Fourier transform of the velocity $\beta(t)$. We introduce the abbreviation

$$\mathcal{J}_{\omega_{\boldsymbol{k}},\omega_{\boldsymbol{k}'}}([\beta(t)], t, t_0) = \int_{t_0}^{t} \mathrm{d}\tau \, \beta(\tau) \mathrm{e}^{\mathrm{i}(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})(\tau - t_0)}. \tag{3.4}$$

By calculating the two matrix elements of equation (3.3), we are led to a striking connection between the transition amplitudes of two-photon states and the expectation values of the force operator; we find

$$c_{\boldsymbol{k},\boldsymbol{k}'} = -\frac{1}{\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'}} \langle \boldsymbol{k}, \boldsymbol{k}'; \xi | F_0 | 0; \xi \rangle \mathcal{J}_{\omega_{\boldsymbol{k}},\omega_{\boldsymbol{k}'}}([\beta(t)], t, t_0). \tag{3.5}$$

This relation, which was first found in [6], reveals the intrinsic interrelation between the photon creation and the radiation pressure of the field. In fact, the imbalance of the fluctuations of the radiation pressure on a non-uniformly moving dielectric body is the underlying cause for the emission of photons.

Finally, we evaluate the matrix element of F_0 by using the field expansion equation (2.9*a*) and the eigenmode functions equation (2.10); after some calculation we obtain

$$c_{\boldsymbol{k},\boldsymbol{k}'} = \mathcal{O}(\boldsymbol{k}, \boldsymbol{k}', n) \mathcal{J}_{\omega_{\boldsymbol{k}},\omega_{\boldsymbol{k}'}}([\beta(t)], t, t_0) \delta(\boldsymbol{k}_{\parallel} + \boldsymbol{k}'_{\parallel}) \tag{3.6}$$

where we have defined the factor of angular distribution $\mathcal{O}(\boldsymbol{k}, \boldsymbol{k}', n)$ by

$$\mathcal{O}(\boldsymbol{k}, \boldsymbol{k}', n) = \frac{(n^2 - 1)}{4\pi n^2} \frac{1}{(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'}) \sqrt{\omega_{\boldsymbol{k}} \omega_{\boldsymbol{k}'} \varepsilon_{\mathrm{in}} \varepsilon'_{\mathrm{in}}}} T^*_{\boldsymbol{k}} T^*_{\boldsymbol{k}'} \{ k_1^{\mathrm{tr}*} k_1'^{\mathrm{tr}*} - \boldsymbol{k}_{\parallel} \cdot \boldsymbol{k}'_{\parallel} \}. \tag{3.7}$$

This expression for the probability amplitudes, which is the central result of the present section, governs the spectral distribution of the emitted two-photon states and provides the means for calculating the radiated energy and the radiation-reaction force.

4. The two-photon spectrum and its angular distribution

The factor $\delta(\boldsymbol{k}_{\parallel} + \boldsymbol{k}'_{\parallel})$ in equation (3.6), which comes from the conservation of momentum parallel to the surface, causes the two photons in each excited two-photon state $|\boldsymbol{k}, \boldsymbol{k}'; \xi\rangle$ to be emitted with equal and opposite surface components of the momentum, i.e. $\boldsymbol{k}_{\parallel} = -\boldsymbol{k}'_{\parallel}$. This means that the planes of incidence of the two photons coincide. However, since their surface components $\boldsymbol{k}_{\parallel}$ and $\boldsymbol{k}'_{\parallel}$ point into opposite directions, their corresponding angles of incidence α and α' are defined as oppositely oriented rotations of the normal vector $\boldsymbol{n}$ in this plane (cf figure 1). These two angles of incidence are related through

$$k_1 \tan \alpha = k'_1 \tan \alpha'. \tag{4.1}$$

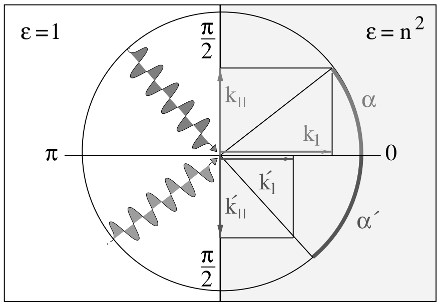

Figure 1. Angles of incidence α and α' of a two-photon state with photons $\boldsymbol{k}$ and $\boldsymbol{k}'$, drawn in the plane of incidence.

Using this relation and recalling equation (2.16), $k_1^{\mathrm{tr}} = k_1\mathcal{C}(\alpha)$, we rewrite the factor of angular distribution in terms of the angles of incidence and find that its modulus square is given by

$$|\mathcal{O}(\alpha,\alpha',n)|^2 = \left(\frac{n^2-1}{\pi n^2}\right)^2 \frac{|\mathcal{D}(\alpha,\alpha')|^2}{\sqrt{\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}\varepsilon_{\boldsymbol{k}'}^{\mathrm{in}}}} \frac{\sin\alpha\sin\alpha'\cos^2\alpha\cos^2\alpha'}{\left(\frac{\sin\alpha}{\sqrt{\varepsilon_{\boldsymbol{k}'}^{\mathrm{in}}}} + \frac{\sin\alpha'}{\sqrt{\varepsilon_{\boldsymbol{k}}^{\mathrm{in}}}}\right)^2}. \tag{4.2}$$

Here we have introduced the complex auxiliary function

$$\mathcal{D}(\alpha,\alpha') \equiv \frac{\mathcal{C}^*(\alpha)\mathcal{C}^*(\alpha') + \tan\alpha\tan\alpha'}{(1+\mathcal{C}^*(\alpha))(1+\mathcal{C}^*(\alpha'))}. \tag{4.3}$$

Hence it becomes evident that the factor of angular distribution is really only a function of the angular configuration of the two-photon state and of the refractive index n of the medium; neither the orientation of the plane of incidence nor the total energy of the photon pair $(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})$ enter this expression. Therefore the expression in equation (3.6) in fact represents a separation of the emission amplitude into a product of two functions of which only the finite-interval Fourier transform of the velocity function β depends on the energy $(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})$ of the two-photon state. In other words, the angular distribution of the emitted photons is governed solely by the factor of angular distribution and is the same for all energies $(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})$ of possible two-photon states.

In order to demonstrate the sensitive dependence of the angular distribution of the emitted photons on the refractive index n of the dielectric medium, we plot the modulus square of the factor of angular distribution for three typical values of n, $n = 1.01$, $n = 1.4$, and $n = 10$; figures 2–4 show the corresponding surface and contour plots. Since we describe a photon emission process, we have plotted the function in terms of ‘out’ states while the mode functions of section 2 represented ‘in’ states which we chose there for their intuitiveness. As the wave equation is invariant under time reversal, ‘in’ and ‘out’ states are two completely equivalent sets of solutions; one emerges from the other by complex conjugation. Note that in the ‘out’-state picture, where the excitation of a two-photon state $|\boldsymbol{k},\boldsymbol{k}'\rangle$ is interpreted as the emission of two photons in the direction of the wavevectors $\boldsymbol{k}$ and $\boldsymbol{k}'$, the critical angle lies in the interval $[0,(\pi/2)]$.

Concentrating first on the contour plot for $n = 1.4$ (figure 3), we can clearly distinguish nine distinct regions in the plot which correspond to different combinations of the two photons that constitute an excited two-photon state. For example, the region $[(\pi/2),\pi]\times[(\pi/2),\pi]$ corresponds to two-photon states with both photons emitted into the

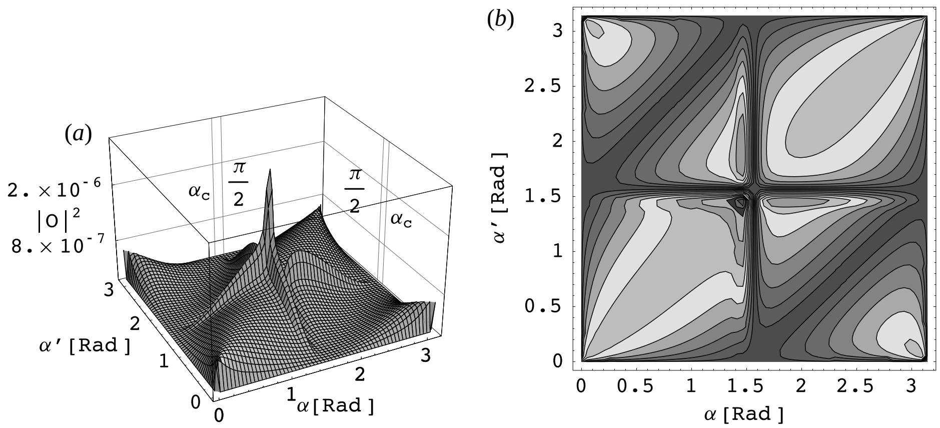

Figure 2. Factor of angular distribution for $n = 1.01$.

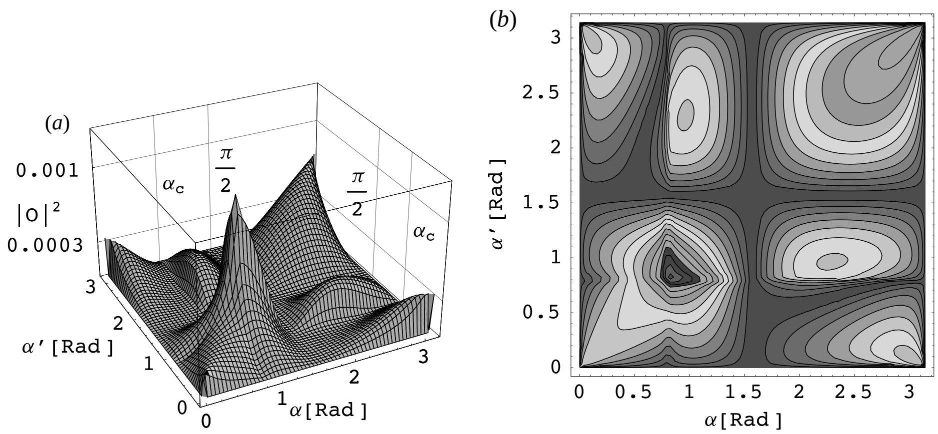

Figure 3. Factor of angular distribution for $n = 1.4$.

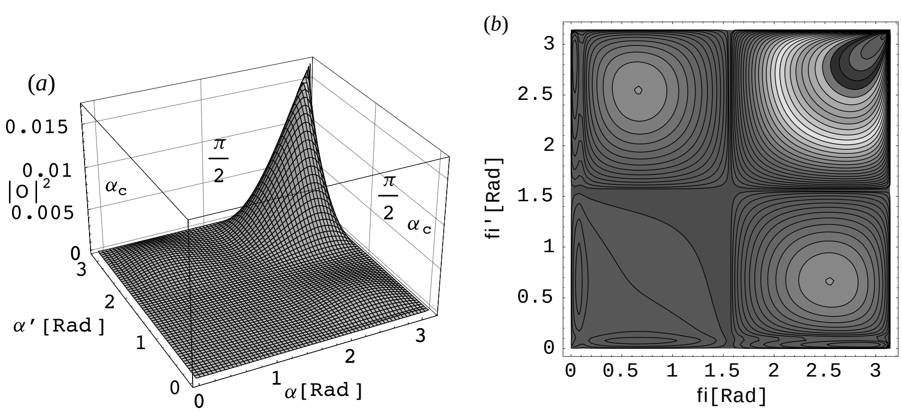

Figure 4. Factor of angular distribution for $n = 10$.

vacuum, and the region $[\alpha_c, (\pi/2)] \times [\alpha_c, (\pi/2)]$ to states with both photons emitted into the medium at an angle of incidence greater than the critical angle which means that they have evanescent components outside the medium. This subregion will play in important role in the next section for the investigation of the radiated energy in the limit of large n.

The surface plot shows that the angular spectrum for $n = 1.4$ is dominated by two peaks that lie on the diagonal, i.e. that correspond to photon states for which both photons have angles of incidence of the same magnitude. One of these peaks, referred to as the peak of critical emission, is centred around angles that correspond to pairs of photons that are emitted into the dielectric medium at the critical angle. The other peak, in the following referred to as the peak of normal emission, corresponds to photon pairs with both photons emitted into the vacuum at normal angle. Whereas this peak is not counterintuitive, the existence of the peak of critical emission could by no means have been anticipated and is a genuinely new feature of quantum radiation from imperfect reflectors. Because it is specific to evanescent wave components it is obviously not retrievable in any one-dimensional calculations.

Turning now to the plots for $n = 10$ (figure 4), we find that the peak of critical emission is no longer visible. The spectrum is now entirely dominated by the peak of normal emission, and the effect calculated is one order of magnitude greater than in the case of $n = 1.4$. This indicates that for high indices of refraction most photons are emitted into the vacuum. In fact, one can show by analytic calculation that the height of the peak of normal emission approaches $(2\pi)^{-2}$ in the perfect-reflector limit and that the ratio between photons emitted normally into the medium and those emitted normally into the vacuum falls as n^{-4} for large values of n. This result agrees with the calculations of [6] for a one-dimensional half-space, and this is what we expect to happen for large n, because the dielectric interface is then almost perfectly reflecting and the propagation of photons inside the medium is strongly suppressed, so that all photons should be radiated into the vacuum.

Looking at the plots for $n = 1.01$ (figure 2), i.e. the case of a very dilute medium, we find that the overall effect has become very small. Of course, any other observation in this limit would be very disconcerting since all quantum radiation must vanish completely for $n \to 1$ where the medium becomes transparent. Interestingly, we find that the peak of critical emission remains a dominant feature of the spectrum even for a very dilute medium. A brief calculation reveals that the relative height of this peak in comparison with that of normal emission tends towards its maximum value of 16 in the vacuum limit of $n \to 1$. Hence this peak is an intrinsic property of the system which is present however close $n > 1$ might be to unity. It cannot be obtained by other approximation methods for dilute media that do not incorporate evanescent wave components, in particular not by the Born approximation.

5. Radiated energy and radiation-reaction force

5.1. Generalities

The probability of having excited a two-photon state $|\boldsymbol{k}, \boldsymbol{k}'; \xi\rangle$ by time t is given by $|c_{\boldsymbol{k},\boldsymbol{k}'}(t)|^2$, so that the total energy $\mathcal{E}_{\text{total}}(t)$ that is radiated into the photon field during the time interval $[t_0, t]$ reads

$$\mathcal{E}_{\text{total}}(t) = \tfrac{1}{2} \int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}' \, (\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'}) |c_{\boldsymbol{k},\boldsymbol{k}'}(t)|^2 \tag{5.1}$$

where the factor $\frac{1}{2}$ prevents double counting of identical photon states in the integration over all $\boldsymbol{k}$ and $\boldsymbol{k}'$. In other words, by total radiated energy we mean the combined energy of all excitations of the field that are on average induced by the motion of the dielectric between the times t_0 and t. The total radiation-reaction force $\mathcal{F}_{\text{total}}(t)$ is defined simply as the expectation value of the force operator F_0, i.e.

$$\mathcal{F}_{\text{total}}(t) = \langle\psi(t)|F_0|\psi(t)\rangle. \tag{5.2}$$

Because $\mathcal{E}_{\text{total}}(t)$ as well as $\mathcal{F}_{\text{total}}(t)$, scale with the size of the $(d-1)$-dimensional surface area $S = L^{(d-1)}$ of the dielectric interface, both of them need to be normalized with respect to the unit area of this surface. Hence we introduce the normalized radiated energy $\mathcal{E}(t)$ and the normalized radiation-reaction force $\mathcal{F}(t)$ by

$$\mathcal{E}(t) = \frac{\mathcal{E}_{\text{total}}(t)}{L^{(d-1)}} \qquad \text{and} \qquad \mathcal{F}(t) = \frac{\mathcal{F}_{\text{total}}(t)}{L^{(d-1)}}. \tag{5.3}$$

Using the identity

$$2\pi\delta(0) = \int_{-\infty}^{\infty} \mathrm{d}x\,\{\mathrm{e}^{-\mathrm{i}xk}\}_{k=0} = \int_{-\infty}^{\infty} \mathrm{d}x = L \tag{5.4}$$

where L is the (infinite) size of our quantization box, we then find for the radiated energy and the radiation-reaction force to first order in β

$$\mathcal{E}(t) = \frac{1}{2}\int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}'\,|\mathcal{J}_{\omega_{\boldsymbol{k}},\omega_{\boldsymbol{k}'}}(t)|^2 \frac{(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})}{(2\pi)^{(d-1)}} |\mathcal{O}(\boldsymbol{k},\boldsymbol{k}',n)|^2 \delta(\boldsymbol{k}_\parallel + \boldsymbol{k}'_\parallel) \tag{5.5}$$

$$\mathcal{F}(t) = -\frac{1}{2}\int \mathrm{d}\boldsymbol{k} \int \mathrm{d}\boldsymbol{k}'\,2\mathrm{Re}\,\{\mathcal{J}_{\omega_{\boldsymbol{k}},\omega_{\boldsymbol{k}'}}(t)\mathrm{e}^{-\mathrm{i}(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})(t-t_0)}\} \frac{(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})}{(2\pi)^{(d-1)}} \times |\mathcal{O}(\boldsymbol{k},\boldsymbol{k}',n)|^2 \delta(\boldsymbol{k}_\parallel + \boldsymbol{k}'_\parallel). \tag{5.6}$$

In order to derive the second of these equations, we have neglected the zeroth-order contribution $\langle 0;\xi|F_0|0;\xi\rangle$ to the matrix element $\langle\psi(t)|F_0|\psi(t)\rangle$, because it would cancel with the corresponding contribution from the opposite interface for any realistic body. This is in fact the only occasion where consistency requires a correction by hand to the model of a semi-infinite half-space. Equations (5.5) and (5.6) entail the important relation

$$\mathcal{E}(t) = -\int_{t_0}^{t} \mathrm{d}\tau\,\beta(\tau)\mathcal{F}(\tau) \tag{5.7}$$

which shows that the energy radiated into the field equals the negative work done on the dielectric body by the force $\mathcal{F}(t)$. This means that the radiation-reaction force $\mathcal{F}(t)$ is a dissipative force which acts as 'friction' on the moving dielectric body; thus the body dissipates part of its kinetic energy into the photon field.

By change of variables we transform the k-space integrals of equations (5.5) and (5.6) into independent integrals over the two angles of incidence α and α' and over the energy $u = \omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'}$ of the two-photon states. Next we define the emissivity factor

$$\mathcal{R}(n,d) = \frac{\pi^{(d-1)/2}}{(2\pi)^{(d-1)}\Gamma(\frac{d-1}{2})}\left(\frac{n^2-1}{\pi n^2}\right)^2 \times \int_0^\pi \mathrm{d}\alpha \int_0^\pi \mathrm{d}\alpha'\,\frac{|\mathcal{D}(\alpha,\alpha')|^2}{\sqrt{\varepsilon_{\boldsymbol{k}}^{\text{in}}\varepsilon_{\boldsymbol{k}'}^{\text{in}}}} \frac{\sin^d\alpha\,\sin^d\alpha'\cos^2\alpha\cos^2\alpha'}{\left(\frac{\sin\alpha}{\sqrt{\varepsilon_{\boldsymbol{k}'}^{\text{in}}}} + \frac{\sin\alpha'}{\sqrt{\varepsilon_{\boldsymbol{k}}^{\text{in}}}}\right)^{(d+3)}} \tag{5.8}$$

where Γ is the standard Gamma function and $\mathcal{D}(\alpha,\alpha')$ is given in equation (4.3). This abbreviation enables us to rewrite equations (5.5) and (5.6) as

$$\mathcal{E}(t) = \mathcal{R}(n,d)\int_0^\infty \mathrm{d}u\,u^{(d+1)}|\mathcal{J}_u(t)|^2 \tag{5.9}$$

$$\mathcal{F}(t) = -\mathcal{R}(n,d)\int_0^\infty \mathrm{d}u\, u^{(d+1)} 2\mathrm{Re}\,\{\mathcal{J}_u(t)\mathrm{e}^{-\mathrm{i}u(t-t_0)}\} \tag{5.10}$$

which are the central equations of this section; they show explicitly that the integration over all possible two-photon states completely separates into the integration over the angular configurations in the emissivity factor $\mathcal{R}(n,d)$ and the integration over the energy of the photon pairs.

5.2. The emissivity factor $\mathcal{R}(n,d)$

Since we have not been able to do the integrations over α and α' analytically, we have calculated $\mathcal{R}(n,d)$ numerically. We have used a 10-point Gauss–Legendre integration with an accuracy greater than the resolution of the plots. Figure 5 shows the emissivity factor $\mathcal{R}(n,d)$ plotted as function of the refractive index n for $d = 2$, $d = 3$, and $d = 4$. Since algebraically the value of d is not restricted to integers, we can plot this function for continuous d, which is instructive for gaining an overall impression of its behaviour; the corresponding surface plot over n and $2 \leqslant d \leqslant 4$ is shown in figure 6. Figure 5 shows that the emissivity factor diverges in the perfect-reflector limit $n \to \infty$ for $d = 4$, which is an interesting and unexpected result of our calculation. By examining the contributions from the various subintegrals over α and α' that correspond to the nine regions of qualitatively different photon pairs familiar from figure 3, we find that this divergence stems from the contribution of the integral over the region $\alpha, \alpha' \in [(\pi/2), \alpha_c] \times [(\pi/2), \alpha_c]$, i.e. from pairs of totally internally reflected photons. Since $\lim_{n\to\infty} \alpha_c = \pi$, this subintegral comprises all radiation emitted into the medium in the perfect reflector limit $n \to \infty$. A lengthy analysis of the asymptotic behaviour of this subintegral, which involves a change of variables for making the integration limits independent of n, reveals that this contribution to $\mathcal{R}(n,d)$ behaves as $n^{(d-3)}$ in the perfect-reflector limit; hence it diverges for $d > 3$ and asymptotically vanishes for $d < 3$. The reason for the divergence in $d > 3$ is that the phase space for totally internally reflected modes grows exponentially with d but the damping of the emission into these modes in the limit $n \to \infty$ is independent of d. The case $d = 3$ is special in that these two effects balance and the amount of radiation going into the medium approaches a constant in the perfect-reflector limit. This places the result by Barton and North [9], who have have found the same for strictly harmonic motions in three dimensions, into a more general context. In particular, we can certify that this behaviour is independent of the type of motion described by the half-space.

We stress that in our calculation the dependence of the emissivity $\mathcal{R}(n,d)$ of the system on the dimension and on the refractive index is not affected by the particular trajectory that the half-space follows; in other words, the above results are independent of the velocity $\beta(t)$. We also point out that the divergence of the emissivity factor, and hence of the radiated energy, for $d > 3$ makes the physical motivation of more than three-dimensional models of moving perfect mirrors highly questionable.

5.3. Radiation-reaction force and spatial dimension

In the following we wish to explore how the radiation-reaction force $\mathcal{F}(t)$ depends on the spatial dimension d. The key idea in this analysis, which we adopt from [6], is to convert the factor $u^{(d+1)}$ in the integrand of equation (5.10) into a time-derivative of the exponential through the relation

$$u^{(d+1)}\mathrm{e}^{\pm\mathrm{i}u\tau} = (\mp\mathrm{i})^{(d+1)}\left(\frac{\partial}{\partial\tau}\right)^{(d+1)}\mathrm{e}^{\pm\mathrm{i}u\tau}. \tag{5.11}$$

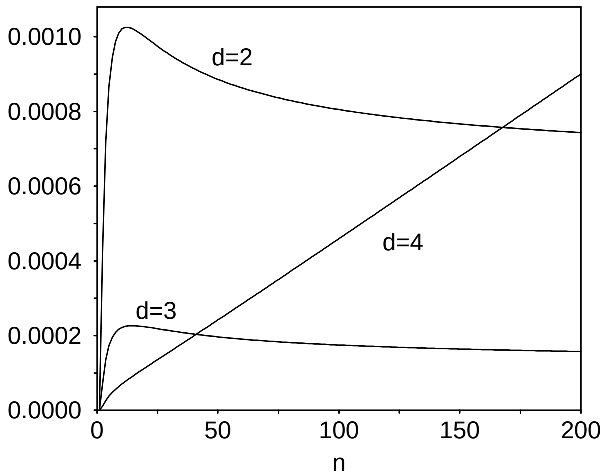

Figure 5. Emissivity factor for $d = 2$, $d = 3$ and $d = 4$.

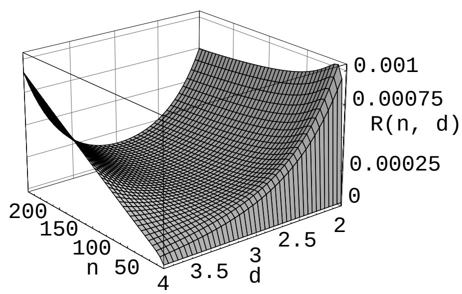

Figure 6. Surface plot of the emissivity factor for continuous d.

In order to avoid mathematically ill-defined expressions, we introduce a cut-off factor $\exp(-\gamma u)$ into the energy integral of equation (5.10). This cut-off is a mathematical regularization device devoid of all physical meaning, and we intend to remove it by taking the no-cut-off limit $\gamma \to 0$ at the end of the calculation when all measurable physical quantities have to emerge cut-off independent. In this limit, all terms vanishing with γ will be discarded while those which remain finite or diverge will be kept.

Thus, using relation (5.11) we find that the radiation-reaction force can be written as

$$\mathcal{F}(t) = -\mathcal{R}(n,d)\int_0^{t-t_0} \mathrm{d}\tau\, \beta(t-\tau)\left(\mathrm{i}\frac{\partial}{\partial\tau}\right)^{(d+1)}\left\{\frac{\gamma((-1)^{(d+1)}+1)+\mathrm{i}\tau((-1)^{(d+1)}-1)}{\gamma^2+\tau^2}\right\}. \tag{5.12}$$

This equation demonstrates the effect of the spatial dimension d on the expression for the radiation-reaction force. For odd spatial dimensions the term in curly brackets reduces to the well known Lorentzian

$$\mathcal{L}_\gamma(\tau) = \frac{\gamma}{\gamma^2+\tau^2} \tag{5.13}$$

but for even spatial dimensions it reads

$$\mathcal{A}_\gamma(\tau) = \frac{\tau}{\gamma^2+\tau^2}. \tag{5.14}$$

In order to show the consequences of this difference, we evaluate the integral for $d = 3$ and $d = 2$ by integration by parts. For this we assume that the velocity $\beta(t)$ of the dielectric body together with all its time derivatives vanishes for $t \leqslant t_0$. This is to say that the motion starts from rest, which is consistent with the assumptions made in the perturbative approach of the previous section, where the initial state of the field prior to perturbation was defined to be the ground state of the unperturbed Hamiltonian H_0.

Beginning with $d = 3$, we find that the force in the no-cut-off limit is given by

$$\mathcal{F}_3(t) = \mathcal{R}(n,3)\left\{\frac{4}{\gamma^3}\frac{\partial}{\partial t}\beta(t) - \frac{2}{\gamma}\frac{\partial^3}{\partial t^3}\beta(t) - \pi\frac{\partial^4}{\partial t^4}\beta(t)\right\}. \tag{5.15}$$

The divergences $(1/\gamma)$ and $(1/\gamma^3)$ appearing in this expression have no physical consequences, because, just as in standard quantum electrodynamics, they can be absorbed into the renormalization of system parameters. In quantum electrodynamics, the coupling of the photon to the electron field leads to a renormalization of the electron mass. Analogously, in our model the coupling of the scalar photon field to the dielectric body through the continuity conditions gives rise to the need for renormalization as well. The term proportional to $(1/\gamma^3)$ simply renormalizes the inertial mass m of the body by

$$\Delta m = \frac{4\mathcal{R}(n,3)}{\gamma^3}. \tag{5.16}$$

We think that the coefficients of higher-order derivatives of the velocity can likewise be renormalized as long as the time-reversal properties of Newton's equation of motion are not affected by the renormalization. To this end we rewrite Newton's second law as

$$F = m_1\frac{\partial\beta}{\partial t} + m_3\frac{\partial^3\beta}{\partial t^3} + \cdots. \tag{5.17}$$

In Newtonian mechanics $m_1 \equiv m$ is the usual inertial mass, and m_3 and all higher coefficients in this equation are zero. From time-reversal symmetry it is clear that the mechanical force F cannot depend on any even time-derivatives of the velocity β because these would give rise to dissipation and hence irreversibility of the equations of motion. However, as far as we know there is no fundamental law that forbids the force to depend on odd time-derivatives higher than first. Although experiments show that the coefficient m_3 is zero within the presently attainable accuracy of measurement, we see no reason why the bare 'third-order mass' m_3^{bare} could not have some other value; this would imply that the observable m_3 is adjusted to zero only by renormalization. Thus, in the same way as the (infinite) bare mass is renormalized to the standard observable mass $m_1 \equiv m$, we propose that the term proportional to $(1/\gamma)$ leads to a renormalization of the bare 'third-order mass' m_3^{bare} by

$$\Delta m_3 = -\frac{2\mathcal{R}(n,3)}{\gamma} \tag{5.18}$$

to $m_3 \approx 0$ for all that has been measured.

Hence, the renormalized radiation-reaction force on a three-dimensional half-space is proportional to the fourth time-derivative of the velocity. It is not difficult to see that this result can readily be generalized to higher odd dimensions d. After renormalization of higher-order masses one obtains that the radiation-reaction force in odd d dimensions is proportional to the $(d+1)$th derivative of the velocity,

$$\mathcal{F}_{\text{odd}}^{(\text{ren})}(t) = -\pi\mathcal{R}(n,3)\frac{\partial^{(d+1)}}{\partial t^{(d+1)}}\beta(t). \tag{5.19}$$

Using relation (5.7), which connects the radiated energy to the radiation-reaction force, the above results for the force also entail expressions for the radiated energy. In three dimensions we find that for motions from rest to rest

$$\mathcal{E}_3 = \pi \mathcal{R}(n,3) \int_{t_0}^{t} \mathrm{d}\tau\, \ddot{\beta}^2(\tau) \tag{5.20}$$

which in the limit $n \to \infty$ agrees with the result of [17]. In arbitrary odd dimensions the radiated energy for trajectories from rest to rest reads

$$\mathcal{E}_{\mathrm{odd}} = \pi \mathcal{R}(n,d) \int_{t_0}^{t_u} \mathrm{d}\tau \left(\frac{\partial^{(d+1)/2}}{\partial \tau^{(d+1)/2}} \beta(\tau) \right)^2 . \tag{5.21}$$

Turning now to even dimensions, we record the force in two dimensions

$$\mathcal{F}_2(t) = 2\mathcal{R}(n,2) \left\{ \frac{\dot{\beta}(t)}{\gamma^2} - \int_0^{t-t_0} \mathrm{d}\tau \;\dddot{\beta}\,(t-\tau) \frac{\tau}{\tau^2+\gamma^2} \right\}. \tag{5.22}$$

The first term in the curly brackets is again just a mass renormalization term, yet the integral in the second does not give a simple expression for the dissipative force because it does not reduce to an instantaneous function of t in the no-cut-off limit; in general it is not even convergent if γ is taken to zero. A similar behaviour is found in arbitrary even dimensions. It is clear that the simple relation (5.19) cannot hold in even dimensions because then the force it gives is odd under time-reversal and can therefore not be dissipative.

In summary, we have found that in odd spatial dimensions the dissipative radiation-reaction force responds instantaneously to the motion of the dielectric half-space and can be expressed as an even time-derivative of the velocity. In contrast, the behaviour of this dissipative force in even dimensions is much more complex; the dissipative response of the system is sensitive to the entire history of the motion, i.e. the force is non-Markovian. The renormalization of mass terms is inevitable in all dimensions.

We ascribe the convergence problems that arise in the no-cut-off limit for the dissipative force in even dimensions to the idealization of non-dispersive dielectrics made in the present model; in a physically more realistic model the high-frequency transparency of any real material would naturally introduce a cut-off at some finite frequency.

5.4. Radiated energy for trial velocity functions

For the illustration of the dependence of the photon emission on the dynamics of the half-space, we calculate the radiated energy for three specific trial functions of the velocity $\beta(t)$. We consider a Lorentzian peak $\beta_{\mathrm{lor}}(t)$, a harmonic oscillation $\beta_{\mathrm{har}}(t)$, and a hyperbolic tangent $\beta_{\tan}(t)$. To facilitate analytic derivations, we let the time interval of the process considered be infinite, i.e. we set $t_0 = -\infty$ and $t = \infty$. Thus equation (5.9), which gives the radiated energy in arbitrary spatial dimensions d, becomes

$$\mathcal{E} = \mathcal{R}(n,d) \int_0^{\infty} \mathrm{d}u\, u^{(d+1)} \left| \int_{-\infty}^{\infty} \mathrm{d}\tau\, \beta(\tau) \mathrm{e}^{\mathrm{i}u\tau} \right|^2 . \tag{5.23}$$

Starting with the Lorentzian

$$\beta_{\mathrm{lor}}(t) = \frac{1}{\pi} \frac{\frac{1}{\Omega}}{t^2 + \frac{1}{\Omega^2}} \tag{5.24}$$

we calculate

$$|\mathcal{J}_u([\beta_{\text{lor}}])|^2 = \frac{1}{\Omega^2\pi^2}\left|\int_{-\infty}^{\infty}\mathrm{d}\tau\,\frac{\cos(u\tau)}{t^2+\frac{1}{\Omega}}\right|^2 = \frac{1}{\Omega^2\pi^2}|\Omega\pi\,\mathrm{e}^{-u/\Omega}|^2 = \mathrm{e}^{-2u/\Omega} \tag{5.25}$$

and find that the radiated energy is given by

$$\mathcal{E}_{\text{lor}} = \mathcal{R}(n,d)\int_0^{\infty}\mathrm{d}u\,u^{(d+1)}\mathrm{e}^{-2u/\Omega} = \mathcal{R}(n,d)(d+1)!\left(\frac{\Omega}{2}\right)^{(d+2)}. \tag{5.26}$$

Fourier transformation of a harmonic motion of amplitude b_{har} and frequency Ω, $\beta_{\text{har}}(t) = b_{\text{har}}\sin(\Omega t)$, yields

$$|\mathcal{J}_u([\beta_{\text{har}}])|^2 = \pi^2 b_{\text{har}}^2|\delta(\Omega-u)|^2 \tag{5.27}$$

hence the amount of energy radiated per unit time by a harmonically oscillating d-dimensional half-space is

$$\frac{\mathcal{E}_{\text{har}}}{T} = \left(\frac{\pi}{2}\right)\mathcal{R}(n,d)b_{\text{har}}^2\Omega^{(d+1)}. \tag{5.28}$$

This equation permits a direct comparison of our results with those by Barton and North [9] who calculate this energy for $d = 3$ and write down its value in the limit $n \to \infty$; we find complete agreement, even in the numerical prefactors. Hence for the specific case of harmonic motion our approximation method is equivalent to Fermi's golden rule employed in [9].

Finally, for

$$\beta_{\tan}(t) = b_{\text{t}}\tanh(\Omega t) \tag{5.29}$$

we have

$$|\mathcal{J}_u([\beta_{\tan}])|^2 = \frac{b_{\text{t}}^2\pi^2}{\Omega^2}\frac{1}{\sinh^2(\frac{u\pi}{2\Omega})} \tag{5.30}$$

which gives,

$$\mathcal{E}_{\tan} = 4b_{\tan}^2\mathcal{R}(n,d)\Gamma(d+2)\zeta(d+1)\left(\frac{\Omega}{\pi}\right)^d \tag{5.31}$$

where Γ is the Gamma function and ζ the Riemann zeta function. Evaluating these special functions in the case $d = 3$, we obtain the energy radiated by a three-dimensional half-space moving with the velocity $\beta_{\tan}(t)$

$$\mathcal{E}_{\tan,3} = \tfrac{16}{15}b_{\tan}^2\mathcal{R}(n,3)\pi\Omega^3. \tag{5.32}$$

Finally, we would like to emphasize that the photon emission depends on the velocity only through the Fourier transform of $\beta(t)$, i.e. on the frequency spectrum of the motion; in other words, general statements about the dependence of the effect on any instantaneous values of velocity—for example, its maximum—are not meaningful by themselves, but have to be made for a specific family of functions; in particular, the radiated energy is not directly connected to the maximum velocity of the half-space. This is very clearly illustrated by the explicit calculations performed above for the trial functions β_{har} and $\beta_{\tan}$; although the velocity is bounded in each case, the amount of radiated energy can, in theory, be made arbitrarily large by increasing Ω.

6. Summary

In this paper we have calculated the characteristics of dissipative radiation emitted by a non-relativistically moving dielectric half-space of arbitrary refractive index $n > 1$ in arbitrary integer spatial dimensions $d > 1$.

The angular distribution of the radiated photons has been calculated and has been shown to be highly sensitive to the refractive index. We have discovered a peak of critical emission which corresponds to photons that are radiated into the dielectric medium at the critical angle of total internal reflection. This peak is an important feature of the spectrum for refractive indices $n \lesssim 3$, which applies to most realistic refractive media in the visible spectrum. Since it is closely related to the presence of total internal reflection and evanescent wave components, this property of the spectral distribution was found neither in one-dimensional models of quantized fields with moving boundary or continuity conditions, e.g. not in [6], nor in any other higher-dimensional calculations for perfectly reflecting moving mirrors, e.g. not in [18], because these models rule out evanescent wave components by construction.

Furthermore, we have calculated the total radiated energy and the radiation-reaction force on the dielectric body per unit surface area of the dielectric interface. We have explored the fundamental difference between systems of odd and even spatial dimensions in connection with the required time-reversal symmetry of dissipative forces.

In particular, we have shown that while in odd-dimensional systems the renormalized radiation-reaction force is proportional to the instantaneous value of the even $(d+1)$th time-derivative of the velocity, in even spatial dimensions the force is non-Markovian, i.e. the system remains sensitive to its history.

For three test trajectories of the dielectric body the amount of energy radiated per unit surface area has been calculated in arbitrary dimensions. Reduced to three dimensions our results agree with previous calculations [17, 9].

The analytical examination of the emissivity factor $\mathcal{R}(n, d)$ has shown that the amount of dissipated energy of a non-uniformly accelerated dielectric half-space, diverges in the perfect-reflector limit $n \to \infty$ for all spatial dimensions greater than three. Hence for $d > 3$ the physical interpretation of models of moving perfect mirrors is questionable.

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